# MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 9

## 1. Prikry forcing

Recall that a cardinal $\kappa$ is measurable if there is a normal, $\kappa$-complete, nonprincipal ultrafilter on $\kappa$. I.e. an ultrafilter $U \subset \mathcal{P}(\kappa)$ such that $U$ is closed under intersection of less than $\kappa$ many sets, and for every $\langle A_\alpha \mid \alpha < \kappa\rangle$, with each $A_\alpha \in U$, the diagonal intersection $\triangle_{\alpha<\kappa} A_\alpha := \{\beta < \kappa \mid \beta \in \triangle_{\alpha<\beta} A_\alpha\} \in U$.

We call such a $U$, a *normal measure* on $\kappa$ and sets in $U$ are called *measure one sets.*

**Lemma 1.** *Suppose that $U$ is a normal measure on $\kappa$, $A \in U$, and $F : A^{<\omega} \to \tau$ for some $\tau < \kappa$. Then there is $B \subset A$, $B \in U$, which is homogeneous for $F$. I.e. for all $n$, $F \restriction [B]^n$ is constant.*

*Proof.* this will be a future homework problem. $\square$

Let $\kappa$ be a measurable cardinal and $U$ be a normal measure on $\kappa$. The Prikry poset, $\mathbb{P}$ consists of pairs $\langle s, A\rangle$, where $s$ is a finite sequence of ordinals in $\kappa$ and $A \in U$. $\langle s_1, A_1\rangle \leq \langle s_0, A_0\rangle$ iff:

- $s_0$ is an initial segment of $s_1$.
- $s_1 \setminus s_0 \subset A_0$,
- $A_1 \subset A_0$.

Given a condition $p = \langle s, A\rangle$, we say that $s$ is the *stem* of $p$.

Let $G$ be $\mathbb{P}$-generic over $V$. Set $s^* = \bigcup\{s \mid (\exists A)\langle s, A\rangle \in G\}$.

**Lemma 2.** *$s^*$ is an $\omega$-sequence cofinal in $\kappa$. And so, in $V[G]$, $\mathrm{cf}(\kappa) = \omega$.*

*Proof.* Suppose that $\alpha < \kappa$. We claim that the set

$$D = \{\langle s, A\rangle \mid \alpha \leq \max(s)\}$$

is dense. For if $\langle s, A\rangle \in \mathbb{P}$, then let $\beta \in A, \beta > \alpha$ with $\max(s) < \beta$. Then $\langle s \cup \{\beta\}, A\rangle \in D$.

So let $\langle s, A\rangle \in D \cap G$. Then $\alpha < \max(s)$; i.e. for $\beta = \max(s) \in s^*$, $\alpha \leq \beta$. It follows that $s^*$ is cofinal in $\kappa$.

For any two $\langle s, A\rangle$, $\langle t, B\rangle$ in $G$, by taking a common extension we see that either $s$ is an initial segment of $t$ or vice versa. Then, for any $\alpha \in s^*$, if $\langle s, A\rangle \in G$ is such that $\alpha \in s$, we have that $s^* \cap \alpha = s \cap \alpha$, which is finite. It follows that $o.t.(s^*) = \omega$ $\square$

**Lemma 3.** *$\mathbb{P}$ has the $\kappa^+$ chain condition.*

*Proof.* Any two conditions with the same stem $\langle s, A\rangle$, $\langle s, B\rangle$ are compatible, since $\langle s, A \cap B\rangle$ is a common extension. Suppose that $\mathcal{A} \subset \mathbb{P}$ is a maximal antichain. Then conditions in $\mathcal{A}$ have different stems. I.e. the cardinality of $\mathcal{A}$ is at most the number of possible stems, which is $\kappa^{<\omega} = \kappa$. □

**Corollary 4.** *$\mathbb{P}$ preserves cardinals greater than or equal to $\kappa^+$.*

Next we have to worry about preservation of cardinals up to $\kappa$. Note that this forcing is not even countable closed. It has, however, the following key property:

**Lemma 5.** *(The Prikry property) Suppose that $\langle s, A\rangle \in \mathbb{P}$ and $\phi$ is a sentence in the forcing language. Then there is a condition $\langle s, B\rangle \leq \langle s, A\rangle$ such that $\langle s, B\rangle$ decides $\phi$ (i.e. $\langle s, B\rangle \Vdash \phi$ or $\langle s, B\rangle \Vdash \neg\phi$).*

*Proof.* Fix $\langle s, A\rangle \in \mathbb{P}$ and $\phi$. Define $F : A^{<\omega} \to 3$ as follows: for $t \in A^{<\omega}$,

(1) if $s^\frown t$ is a stem and there is $B \subset A$, such that $\langle s^\frown t, B\rangle \Vdash \phi$, then $F(t) = 0$;
(2) if $s^\frown t$ is a stem and there is $B \subset A$, such that $\langle s^\frown t, B\rangle \Vdash \neg\phi$, then $F(t) = 1$;
(3) otherwise, $F(t) = 2$.

Note that since conditions with the same stem are compatible, it is impossible to fall into both cases 1 and 2. So, $F$ is well defined.

By Lemma 1, there is $B \subset A$, $B \in U$, for which $F$ is homogeneous. We claim that $\langle s, B\rangle$ decides $\phi$. Otherwise there are conditions $r = \langle t_r, B_r\rangle, q = \langle t_q, B_q\rangle$, $r, q \leq \langle s, B\rangle$, such that $r \Vdash \phi$ and $q \Vdash \neg\phi$. By extending these if necessary, we may assume that $|t_q| = |t_r| = k > |s|$. Let $n = k - |s|$. Then $F(t_q) \neq F(t_r)$. Contradiction with $F$ constant on $[B]^n$. □

**Lemma 6.** *$\mathbb{P}$ does not add new bounded subsets of $\kappa$.*

*Proof.* Suppose that $G$ is $\mathbb{P}$-generic and $a \in V[G]$ is a bounded subset of $\kappa$. I.e. for some $\lambda < \kappa$, $a \subset \lambda$. Let $p = \langle s, A\rangle \Vdash \dot{a} \subset \lambda$. For every $\alpha < \lambda$, let $\langle s, A_\alpha\rangle$ decide "$\alpha \in \dot{a}$". Let $q = \langle s, \bigcap_{\alpha<\lambda} A_\alpha\rangle$, and $b = \{\alpha < \lambda \mid q \Vdash \alpha \in \dot{a}\}$. Then $q \Vdash b = \dot{a}$, i.e. $q$ forces that $\dot{a}$ is in $V$.

By density it follows that there is such a condition $q$ is $G$. So $a \in V$. □

**Corollary 7.** *$\mathbb{P}$ preserves cardinals up to and including $\kappa$.*

*Proof.* Suppose otherwise. Let $G$ be $\mathbb{P}$-generic over $V$. Let $\lambda \leq \kappa$ be the least cardinal collapsed. Since, a limit of cardinals is always a cardinal, and $\kappa$ is limit, it follows that $\lambda < \kappa$, and $\lambda$ is regular in $V$.

Then in $V[G]$, there is some cardinal $\tau < \lambda$, and a confinal function $f : \tau \to \lambda$. (Here $\tau$ is a cardinal in both $V$ and $V[G]$). Then $a := \mathrm{ran}(f)$ is a bounded subset of $\kappa$, so by the above $a \in V$. But then we have in that $V$, $a$ is a cofinal subset of the regular cardinal $\lambda$ with $|a| = \tau$. Contradiction. □

**Corollary 8.** *$V$ and $V[G]$ have the same cardinals.*

## 2. An application: violating SCH

Recall that $Add(\kappa, \lambda)$ is the poset of partial functions from $\kappa \times \lambda$ to $\{0, 1\}$ of size less than $\kappa$, ordered by extension. For a regular cardinal $\kappa$, forcing with $Add(\kappa, \lambda)$ adds $\lambda$ many new subsets of $\kappa$ and preserves cardinals. So, it is fairly easy to increase the powerset of a regular cardinal, and we can do it in ZFC. But for singular $\kappa$, it is much more difficult.

**Definition 9.** *Let $\kappa$ be a singular cardinal. The singular cardinal hypothesis, SCH, holds at $\kappa$, if $2^{\mathrm{cf}(\kappa)} < \kappa$ implies $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. If $\kappa$ is strong limit, that is equivalent to saying that $2^\kappa = \kappa^+$.*

GCH implies SCH. However, we can't use the Cohen poset to violate SCH: suppose $\kappa$ is singular, and we force with $Add(\kappa, \kappa^{++})$. This will add new subsets, but it is no longer $\kappa$-closed. And actually this poset will collapse $\kappa$.

To violate SCH we need a different strategy. The basic idea is to start with some large, and so regular, cardinal $\kappa$, force with $Add(\kappa, \kappa^{++})$, and then singularize $\kappa$. We make use of the following fact:

*Fact* 10. Assuming enough large cardinals, we can arrange that in $V$, $\kappa$ is measurable and $2^\kappa = \kappa^{++}$.

**Theorem 11.** *Assuming enough large cardinals, there is a forcing extension in which SCH fails.*

*Proof.* Let $V$ be such that $\kappa$ is measurable and $2^\kappa = \kappa^{++}$ and let $\mathbb{P}$ be the Prikry poset. Let $G$ be $\mathbb{P}$-generic. Then in $V[G]$, $\kappa$ is singular with cofinality $\omega$, and we still have $2^\kappa = \kappa^{++}$. Moreover, since $\mathbb{P}$ does not add any bounded subsets of $\kappa$, $\kappa$ is strong limit in $V[G]$. It follows that SCH fails at $\kappa$.

$\square$

*Remark* 1. The optimal hypothesis is a measurable $\kappa$ of Mitchel order $\kappa^{++}$.

## 3. Characterization of genericity for $\mathbb{P}$

Next we will show that using a slight strengthening of the Prikry property, we can isolate a fairly simple necessary and sufficient condition for an object $G \subset \mathbb{P}$ to be a generic filter for $\mathbb{P}$ over $V$.

**Theorem 12.** *Suppose $U$ is a normal measure on $\kappa$, and $\mathbb{P}$ is the Prikry poset defined with respect to $U$. Let $s^* = \langle \alpha_n \mid n < \omega \rangle$ be an increasing sequence through $\kappa$ and $G := \{\langle s, A \rangle \mid (\exists n)(s = \langle \alpha_0, ..., \alpha_{n-1} \rangle,$ and $\forall k \geq n, \alpha_k \in A)\}$. Then $G$ is a generic filter for $\mathbb{P}$ over $V$ iff for every $A \in U$, for all large $n$, $\alpha_n \in A$.*

For the easier direction, suppose that $G$ is a generic filter, and let $A \in U$. The set $D := \{\langle s, B \rangle \in \mathbb{P} \mid A \subset B\}$ is dense, so there is $\langle s, B \rangle \in G \cap D$. That means that for all large $n$, $\alpha_n \in B \subset A$.

Now suppose that $\langle \alpha_n \mid n < \omega \rangle$ is increasing, such that for every $A \in U$, for all large $n$, $\alpha_n \in A$. Let $G := \{\langle s, A \rangle \mid (\exists n)(s = \langle \alpha_0, ..., \alpha_{n-1} \rangle,$ and $\forall k \geq n, \alpha_k \in A)\}$. We want to show that $G$ is a generic filter.

$G$ is upwards closed by definition. Now suppose that $\langle s_1, A_1\rangle, \langle s_2, A_2\rangle$ are both in $G$. Let $n_1, n_2$ be such that $s_1 = \langle \alpha_0, ..., \alpha_{n_1}\rangle$ and $s_2 = \langle \alpha_0, ..., \alpha_{n_2}\rangle$. Say $n_1 \leq n_2$. Then $s_1$ is an initial segment of $s_2$ and $s_2 \setminus s_1 \subset A_1$. It follows that $\langle s_2, A_2 \cap A_1\rangle$ is a common extension in $G$. So $G$ is a filter.

To show genericity, suppose that $D' \subset \mathbb{P}$ is a dense set. Let $D = \{p \mid (\exists q \in D')p \leq q\}$, i.e. the downward closure of $D'$. Since $G$ is a filter, it is enough to show that $G \cap D \neq \emptyset$. Such a set $D$ is called dense open. We will use the following strengthening of the Prikry lemma:

**Lemma 13.** *For every dense open $D \subset \mathbb{P}$, for every stem $t$ (i.e. $t \in \kappa^{<\omega}$ is an increasing sequence), there is some $n$ and $A \in U$, with $A \subset \kappa \setminus \max(t)+1$, such that for every increasing $s \in [A]^n$, $\langle t^\frown s, A \setminus \max(s) + 1\rangle \in D$.*

*Proof.* Fix $D$ and $t$. For every $s$, such that $t^\frown s$ is a finite increasing sequence, let $A_s \in U$, $A_s \subset \kappa \setminus \max(s) + 1$, be such that $\langle t^\frown s, A_s\rangle \in D$ if such a set exists. Otherwise set $A_s = \kappa \setminus \max(s) + 1$. Let $B = \triangle_s A_s := \{\alpha \mid \alpha \in \bigcap_{\max(s)<\alpha} A_s\}$. This is a slight modification of diagonal intersection, and with some work, by normality of $U$, we get $B \in U$.

Let $F : B^{<\omega} \to \{0, 1\}$ be $F(s) = 0$ if $\langle t^\frown s, A_s\rangle \in D$, and $F(s) = 1$ otherwise. By lemma 1, let $A \in U$ be a homogeneous set for $F$.

Since $D$ is dense, let $\langle t^\frown h, A'\rangle \leq \langle t, A\rangle$ be such that $\langle t^\frown h, A'\rangle \in D$. Set $n = |h|$. We claim that $A, n$ are as desired. Suppose that $s \in [A]^n$ is an increasing sequence.

**Claim 14.** $\langle t^\frown s, A_s\rangle \in D$.

*Proof.* Since $F \restriction [A]^n$ is constant, $h \in [A]^n$ and $F(h) = 0$, we have that $F(s) = 0$ □

By the definition of diagonal intersection, for any $\alpha \in A \subset B$, if $\alpha > \max(s)$, then $\alpha \in A_s$. Then $\langle t^\frown s, A \setminus \max(s) + 1\rangle \leq \langle t^\frown s, A_s\rangle \in D$. So, $\langle t^\frown s, A \setminus \max(s) + 1\rangle \in D$.

□

For all stems $t$, fix $n_t$ and $A_t$ as in the conclusion of the above lemma. Let $A = \triangle A_t := \{\alpha \mid \alpha \in \bigcap_{\max(t)<\alpha} A_t\} \in U$.

Let $n$ be such that for all $k \geq n$, $\alpha_k \in A$. Let $t = \langle \alpha_0, ..., \alpha_{n-1}\rangle$. Let $s = \langle \alpha_n, ..., \alpha_{n+n_t-1}\rangle$. Then by the definition of diagonal intersection, $s \in [A_t]^{n_t}$, so $\langle t^\frown s, A_t \setminus \max(s) + 1\rangle \in D$. And since $\langle t^\frown s, A \setminus \max(s) + 1\rangle \leq \langle t^\frown s, A_t \setminus \max(s) + 1\rangle$, we have that $\langle t^\frown s, A \setminus \max(s) + 1\rangle \in D$. But also, by definition of $G$, we have that $\langle t^\frown s, A \setminus \max(s) + 1\rangle \in G$. So, $G \cap D \neq \emptyset$.